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Understanding the damping of a quantum harmonic oscillator coupled to a two-level system using analogies to classical friction

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A quantum harmonic oscillator coupled to a two-level system provides a tractable model of many physical systems from atoms in an optical cavity, to superconducting qubits coupled to an oscillator, to quantum dots in a photonic crystal. When the system experiences damping, the problem becomes considerably more complicated. We demonstrate how to gain insight by drawing analogies to classical damping. Specifically, we show how a quantum harmonic oscillator coupled to a damped two-level system can display two types of frictional behavior, corresponding to classical motion in a fluid and motion on a rough surface. We further show that this system can be tuned continuously between these two regimes.
[http://dx.doi.org/10.1119/1.4735707]

I. INTRODUCTION

The quantum harmonic oscillator is one of the most important models in physics; its elaborations are capable of describing an astonishing breadth of physical phenomena. Jaynes and Cummings studied a quantum harmonic oscillator coupled to a two-level system,[1] which is used to model systems like atoms in an optical cavity,[2] superconducting qubits coupled to a superconducting resonator,[3] or quantum dots in a photonic crystal.[4] To form an accurate picture of some systems, damping must be added to the Jaynes–Cummings model, which makes it considerably more difficult to analyze.

In this article, we consider the case in which only the two-level system (TLS) is damped; the quantum harmonic oscillator interacts with a thermal reservoir but only through the TLS as indicated in Fig. 1. To simplify the dynamics, we assume the oscillator–TLS coupling is weaker than the TLS–bath interaction, the so-called regime of weak coupling. Starting with some quanta in the oscillator and with the TLS in its ground state, we investigate how the oscillator loses energy. We show that if relatively few quanta are initially placed in the oscillator, the TLS provides an effective link to the reservoir, and the oscillator loses energy exponentially with time. On the other hand if the oscillator initially contains a large number of quanta, the TLS link becomes congested, and the oscillator loses energy linearly with time. A crossover regime occurs when the oscillator begins with an intermediate number of quanta.

To gain intuition into the oscillator's behavior, we find it useful to draw an analogy to classical friction, familiar from both undergraduate[5,6] and graduate[7,8] physics education. A popular example of classical frictional dynamics is provided by the damped simple harmonic oscillator equation

$$m_r\ddot{x} + \gamma_w\dot{x} + m_r\omega_r^2 x = 0, \tag{1}$$

where x denotes the position of the oscillator, ω_r and m_r its resonant frequency and mass, respectively, γ_w is the damping constant, and the dots indicate derivatives with respect to time. Loss of energy from the oscillator occurs due to the second term in Eq. (1). This term describes a frictional force whose magnitude is determined by γ_w as well as the instantaneous velocity of the oscillator and whose direction is always opposite the aforementioned velocity. Such a viscous force causes an exponential loss of energy with time; it is typical of any fluid medium that impedes the motion of the oscillator and is sometimes referred to as "wet" or "fluid" friction. A pendulum damped by air is a good example of this type of friction. Our damped quantum oscillator, when it starts with relatively few quanta and loses energy exponentially in time via the TLS, behaves as if subjected to fluid friction.

Another popular example of classical frictional dynamics assumes a kind of frictional force whose direction is always opposite the velocity of an oscillator but whose magnitude is fixed. This sort of friction is sometimes referred to as "sliding" or "dry" friction.[9–13] An equation describing the resulting motion of such an oscillator is

$$m_r\ddot{x} + \gamma_d\,\mathrm{sgn}(\dot{x}) + m\omega_r^2 x = 0, \tag{2}$$

where γ_d has dimensions of force. A mass attached to a spring and oscillating on a rough surface is a good example of this type of friction. The magnitude of the frictional force is fixed at $\gamma_d = \mu m_r g$, where μ is the coefficient of sliding friction and g is the gravitational field strength. The energy loss in this case has a linear time dependence as a result of the damping. Our damped quantum oscillator, when it starts with a relatively large number of quanta and loses energy linearly in time via the TLS, behaves as if subjected to dry friction.

These phenomena were noticed in Ref. 14, which dealt with a more complicated system and only briefly referred to the analogy with classical friction. In this article, we present a simplified version of the earlier system and give three reasons why such an analysis has pedagogical value. First, it provides students of quantum mechanics with a microscopic quantum model on which they may exercise their classical intuition regarding friction. Second, the simple analytic expressions derived in this article enable a complete exploration of fluid and dry friction and allow for a continuous tuning from one type to the other. Last, the model provides a good introduction to the tractable and versatile damped Jaynes–Cummings model.

The rest of the paper is organized as follows. Section II describes a "closed" oscillator–TLS system in quantum mechanical language. Section III couples a bath at zero temperature to the TLS and describes the resulting "open" quantum system using the density matrix. Section IV provides

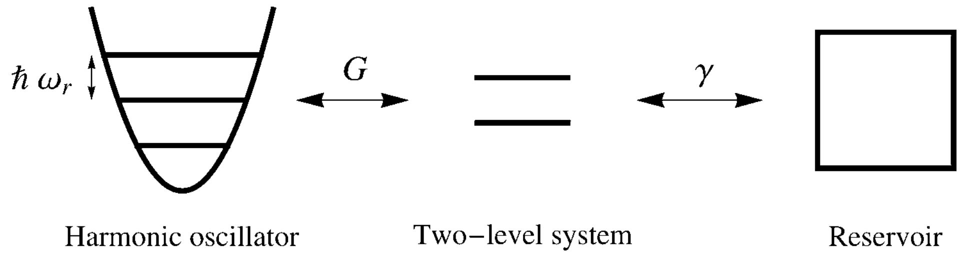

Fig. 1. A schematic of the physical system described in this article. A quantum mechanical harmonic oscillator of frequency ω_r is coupled with a strength G to a two-level system, which in turn is damped at a rate γ by a bath at zero temperature.

approximate analytical results accounting for fluid and dry behavior. Section V suggests some exercises for the reader, and Sec. VI supplies a discussion.

II. HAMILTONIAN FOR THE CLOSED QUANTUM SYSTEM

In this section, we consider only the simple harmonic oscillator connected to the TLS, as shown in Fig. 1, and ignore the bath. A quantum mechanical Hamiltonian that represents the TLS–oscillator system "closed" to the rest of the universe is given by

$$H = \hbar\omega_r a^\dagger a + \frac{\hbar\omega_r}{2}\sigma_z + \hbar G(a\sigma_{10} + a^\dagger\sigma_{01}), \tag{3}$$

where $\hbar$ is Planck's constant (divided by 2π). This is the Jaynes–Cummings Hamiltonian[1] familiar from many textbooks[2] and articles,[15,16] and for which the eigenenergies and eigenstates can be obtained analytically.[2]

The first term in Eq. (3) represents the energy of the harmonic oscillator in terms of the annihilation (a) and creation ($a^\dagger$) operators, which together obey the Bosonic commutation rule

$$[a, a^\dagger] = 1. \tag{4}$$

The second term in Eq. (3) denotes the energy of the TLS. The operator

$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0| \tag{5}$$

is the Pauli z matrix that corresponds to the population difference between the two TLS levels. The third term in Eq. (3) represents the coupling between the oscillator and the TLS, measured by the rate G. The operators

$$\sigma_{10} = |1\rangle\langle 0| \quad \text{and} \quad \sigma_{01} = |0\rangle\langle 1| \tag{6}$$

represent excitation and de-excitation of the TLS, respectively. Thus, the term $a^\dagger\sigma_{01}$ in the Hamiltonian corresponds to a process in which a single quantum is transferred from the TLS to the oscillator and the term $a\sigma_{10}$ corresponds to transfer in the opposite direction.

In writing the coupling term, we have assumed[17]

$$G \ll \omega_r. \tag{7}$$

As long as this inequality is maintained, processes in which the oscillator and the TLS are simultaneously excited ($a^\dagger\sigma_{01}$) or de-excited ($a\sigma_{01}$) occur with low probability and can be justifiably neglected in writing Eq. (3), i.e., they are negligible with respect to any other term in that Hamiltonian. However, when the inequality is violated, these processes can no longer be neglected, and a corresponding term $\hbar G(a^\dagger\sigma_{10} + a\sigma_{01})$ must be added to Eq. (3). Although this term is smaller than the last term of Eq. (3), it is not negligible with respect to the first two terms. The breakdown of the Jaynes–Cummings model and the effects of the additional terms have recently been experimentally observed, for $G/\omega_r \sim 12\%$.[18]

It is important to note that the frequency difference between the lower ($|0\rangle$) and upper ($|1\rangle$) TLS levels has been chosen equal to the frequency of the oscillator ω_r in order to simplify the analysis. This choice maximizes the ability of the TLS to accept quanta of excitation from the oscillator—Hamiltonian evolution can then transfer a quantum initially in the harmonic oscillator into the TLS with unit probability. When the frequencies of the oscillator and the TLS are detuned, the probability of transfer is less than one and the dynamics becomes more complicated.

III. EQUATION FOR THE OPEN QUANTUM SYSTEM

We now consider the coupling of the TLS to a reservoir. A wave function approach no longer suffices because a system interacting with the environment cannot generally be described by a pure state,[19] so a density matrix ρ is required to describe the now "open" system. The density matrix operator for any system may generally be written as

$$\rho = \sum_i P_i |i\rangle\langle i|, \tag{8}$$

where P_i is the probability for the system to be in the state $|i\rangle$. The expectation value of any operator A is given by its trace over the density matrix: $\langle A\rangle = Tr[A\rho]$.

The density matrix for the TLS–oscillator system of Fig. 1 obeys a "master" equation[19] of the form[20]

$$\dot\rho = -\frac{i}{\hbar}[H, \rho] + \frac{\gamma}{2}(2\sigma_{01}\rho\sigma_{10} - \sigma_{10}\sigma_{01}\rho - \rho\sigma_{10}\sigma_{01}), \tag{9}$$

where the square brackets signify a commutator. The first term on the RHS of this equation involves the Hamiltonian of Eq. (3) and accounts for the dynamics internal to the "closed" system discussed in Sec. II.

The rest of the terms on the RHS of Eq. (9) describe the coupling of the system to the bath, where γ represents the rate at which quanta are scattered from the upper TLS level into the bath. This type of term can generally be derived by assuming that the bath couples weakly to the TLS and may thus be treated as a perturbation. The derivation is straightforward, although some additional assumptions and tedious algebra, detailed in Ref. 20, are required. Some intuition regarding Eq. (9) can be gained by imagining that $\rho(t_1)$ takes the form $|\Psi(t_1)\rangle\langle\Psi(t_1)|$ for some system state $|\Psi(t_1)\rangle$ at an instant t_1. Then, the term $\frac{\gamma}{2}(2\sigma_{01}\rho(t_1)\sigma_{10}) = \gamma\sigma_{01}|\Psi(t_1)\rangle\langle\Psi(t_1)|\sigma_{10}$ describes a process in which the lower TLS level in the state $|\Psi(t_1)\rangle$ gains population from the upper TLS level via spontaneous emission. The terms $\frac{\gamma}{2}(-\sigma_{10}\sigma_{01}\rho-\rho\sigma_{10}\sigma_{01})=\frac{\gamma}{2}(-\sigma_{10}\sigma_{01}|\Psi(t_1)\rangle\langle\Psi(t_1)|-|\Psi(t_1)\rangle\langle\Psi(t_1)|\sigma_{10}\sigma_{01})$ ensure that the upper TLS population is reduced accordingly to conserve probability. A more thorough understanding of the master equation is achieved by using it to calculate the dynamical equations for the averages of physically meaningful quantities, as will be done below.

In this article, we will always assume that the oscillator–TLS interaction is smaller than the TLS–bath coupling, i.e.,

$$G < \gamma, \tag{10}$$

so that, as far as the oscillator is concerned, the TLS is more a conduit of energy to the bath than an equal part of the system. In the literature, this is known as the "weak-coupling" regime of the problem.

The full behavior of our model, including the oscillator photon number, the excited and ground state populations of the TLS, as well as correlations between the two systems, can be extracted by numerically solving Eq. (9). Our basic approach to obtain these solutions will be to begin with quanta $\langle n(0)\rangle$ in the oscillator and with the TLS in the ground state $|0\rangle$. Then, as the system evolves, we will observe the number of quanta in the oscillator $\langle n(t)\rangle$ as a function of time, where

$$n(t) = a^\dagger a \tag{11}$$

is the number operator. Since the oscillator energy is given by $\hbar\omega_r\langle n\rangle$, the decay behavior of $\langle n\rangle$ will reveal the type of friction the oscillator experiences.

The method of numerical solution of the master equation of Eq. (9) is described in many references in the literature, the most directly useful being Ref. 21. Our master equation can be recovered from their Eq. (1) by setting $k=0$, and by writing their TLS states $|1\rangle$ and $|2\rangle$ in our notation as $|0\rangle$ and $|1\rangle$, respectively. The numerical results presented in our article can be generated using their Eqs. (17)–(20). However, our initial conditions are instead [with reference to their Eq. (21)]

$$P_n^{(1)} = -P_n^{(2)} = \delta_{n,\langle n(0)\rangle}, \quad P_n^{(3)} = P_n^{(4)} = 0, \tag{12}$$

where $\langle n(0)\rangle$ is the initial number of quanta in the oscillator.

IV. ANALYTICAL SOLUTIONS

In order to deepen our understanding of the problem, we now try to find an approximate analytical solution. In order to do this, we must calculate expectation values of several operators; for example, we will need to calculate $\langle a\rangle$. Remembering that the expectation value of an operator is given by its trace over the density matrix, we can write

$$\begin{aligned}\frac{d}{dt}\langle a\rangle &= \frac{d}{dt}\mathrm{Tr}[a\rho],\\ &= \mathrm{Tr}[a\dot{\rho}],\end{aligned} \tag{13}$$

where in the second line, we have exploited the fact that a does not depend explicitly on time. We can now supply the RHS of Eq. (13) by using Eq. (9) to evaluate the trace. Particularly useful in evaluating the trace is the property that the trace is invariant under cyclic permutations so that

$$\mathrm{Tr}[a\rho a^\dagger] = \mathrm{Tr}[a^\dagger a\rho] = \langle n\rangle, \tag{14}$$

for example. Using the relations in Eqs. (13) and (14), we then find

$$\frac{d}{dt}\langle a\rangle = -i(\omega_r\langle a\rangle + G\langle\sigma_{01}\rangle), \tag{15}$$

$$\frac{d}{dt}\langle\sigma_{11}\rangle = -iG(\langle a\sigma_{10}\rangle - \langle a^\dagger\sigma_{01}\rangle) - \gamma\langle\sigma_{11}\rangle, \tag{16}$$

$$\frac{d}{dt}\langle\sigma_{01}\rangle = -i[\omega_r\langle\sigma_{01}\rangle - G(2\langle a\sigma_{11}\rangle - \langle a\rangle)] - \frac{\gamma}{2}\langle\sigma_{01}\rangle, \tag{17}$$

$$\frac{d}{dt}\langle n\rangle = iG(\langle a\sigma_{10}\rangle - \langle a^\dagger\sigma_{01}\rangle), \tag{18}$$

where

$$\sigma_{11} = |1\rangle\langle 1| \tag{19}$$

is the population operator for the upper TLS level.

Some comments regarding Eqs. (15)–(18) are in order. First, we note from the last term on the RHS of Eq. (16) that $\langle\sigma_{11}\rangle$, the upper state population of the TLS, is damped at the rate γ due to spontaneous emission. Similarly, from Eq. (17), we recognize that $\langle\sigma_{01}\rangle$, the TLS "coherence," is damped at the rate $\gamma/2$, an effect also due to spontaneous emission. The quantity $\gamma/2$ is usually referred to as the dephasing rate of the coherence. Taking the Hermitian conjugate of Eq. (17) also implies that $\langle\sigma_{10}\rangle$ is dephased at the same rate. Last, it is revealing to combine Eqs. (16) and (18) into a single equation reading

$$\frac{d}{dt}[\langle n(t)\rangle + \langle\sigma_{11}\rangle] = -\gamma\langle\sigma_{11}\rangle, \tag{20}$$

which states that the rate of change of excitation in the coupled TLS–oscillator system [given by the left hand side of Eq. (20)] equals the rate at which quanta are emitted into the reservoir by the TLS [given by the right hand side of Eq. (20)].

To proceed further, we make a simplifying assumption regarding the terms in Eqs. (15)–(18) that appear as expectation values of products of operators (such as $\langle a\sigma_{11}\rangle$) and which represent correlations between the oscillator and the TLS. Generally these correlations are important, and decorrelations such as $\langle a\sigma_{11}\rangle \sim \langle a\rangle\langle\sigma_{11}\rangle$ may not be performed. In this case, it can be readily seen that Eqs. (15)–(18) do not form a closed set of equations. However, when the TLS is weakly excited, its correlations with the oscillator are small. Also when the TLS is strongly excited, it is in a fully mixed state. In both cases, its own density matrix is diagonal, independent of the oscillator. We can then write the system density matrix as a product of the oscillator and TLS matrices, implying that the correlations between the two subsystems are negligible. For these two cases, we may decorrelate Eqs. (16)–(18). Further simplification of Eqs. (15)–(18) can be obtained by a change of variables denoted as

$$\langle a \rangle \to \langle a \rangle e^{-i\omega_r t}, \quad \langle \sigma_{10} \rangle \to \langle \sigma_{10} \rangle e^{i\omega_r t}, \tag{21}$$

which also implies

$$\langle a^\dagger \rangle \to \langle a^\dagger \rangle e^{i\omega_r t}, \quad \langle \sigma_{01} \rangle \to \langle \sigma_{01} \rangle e^{-i\omega_r t} \tag{22}$$

and corresponds to a transformation to a frame rotating at the frequency ω_r. Implementing these changes, we obtain from Eqs. (15)–(18),

$$\frac{d}{dt}\langle a \rangle = -iG\langle \sigma_{01} \rangle, \tag{23}$$

$$\frac{d}{dt}\langle \sigma_{11} \rangle = -iG(\langle a \rangle\langle \sigma_{10} \rangle - \langle a^\dagger \rangle\langle \sigma_{01} \rangle) - \gamma\langle \sigma_{11} \rangle, \tag{24}$$

$$\frac{d}{dt}\langle \sigma_{01} \rangle = 2iG\langle a \rangle\langle \sigma_{11} \rangle - ig\langle a \rangle - \frac{\gamma}{2}\langle \sigma_{01} \rangle, \tag{25}$$

$$\frac{d}{dt}\langle n \rangle = iG(\langle a \rangle\langle \sigma_{10} \rangle - \langle a^\dagger \rangle\langle \sigma_{01} \rangle). \tag{26}$$

The inequality of Eq. (10) suggests that the TLS emits energy quickly to return to its steady state. It is therefore proper to consider the steady state solutions to Eqs. (24) and (25) obtained by setting $d\langle \sigma_{11} \rangle/dt$ and $d\langle \sigma_{01} \rangle/dt$ to zero. The result is a rather simple expression for the steady state TLS excitation

$$\langle \sigma_{11} \rangle_s = \frac{1}{2}\frac{R^2(t)}{1+R^2(t)}, \tag{27}$$

which depends on the single dimensionless parameter

$$R(t) = \frac{2\sqrt{2}G\langle n(t) \rangle^{1/2}}{\gamma}. \tag{28}$$

Writing $\langle \sigma_{11} \rangle = \langle \sigma_{11} \rangle_s + \langle \sigma_{11} \rangle_t$, where $\langle \sigma_{11} \rangle_t$ is a transient deviation away from the steady state value, it can be shown mathematically that $\langle \sigma_{11} \rangle_t$ becomes small very quickly, i.e., after a time very short compared to the characteristic timescale of the system dynamics. To obtain the results below, we will thus use the approximation $\langle \sigma_{11} \rangle \simeq \langle \sigma_{11} \rangle_s$.

A. Fluid friction: $R(t) \ll 1$

We now consider the regime in which $R(t) \ll 1$, which can be arranged by starting with a low number of quanta in the oscillator. In this case, Eq. (27) implies $\langle \sigma_{11} \rangle_s \simeq R^2(t)/2$ and using $\langle \sigma_{11} \rangle \simeq \langle \sigma_{11} \rangle_s$ in Eq. (20) yields the simple solution

$$\langle n(t) \rangle = \langle n(0) \rangle e^{-\Gamma t}, \tag{29}$$

where the inverse of the effective decay rate given by

$$\Gamma^{-1} = \frac{\gamma}{4G^2} + \frac{1}{\gamma}. \tag{30}$$

The analytic expression of Eq. (29) compares reasonably well with the numerical solution of Eq. (9), as can be seen from Fig. 2. This figure shows how the number of quanta in the oscillator decreases with time, starting initially with $\langle n(0) \rangle = 2$. The formula of Eq. (29) is straightforward to plot, using the parameters supplied in the caption of Fig. 2.

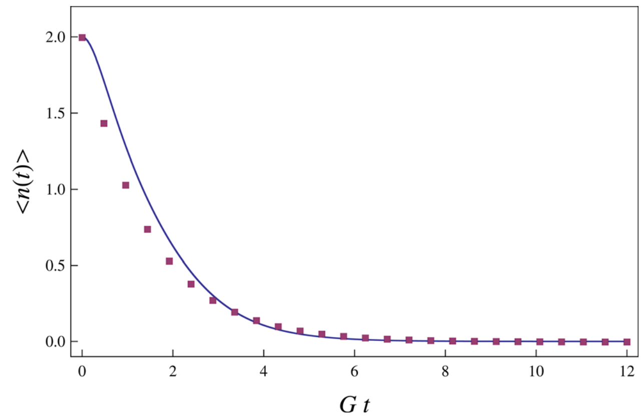

Fig. 2. A graph of oscillator photon number as a function of time in the fluid friction regime. The numerical solution [Eq. (9)] is shown as a solid curve while the analytical solution [Eq. (29)] is plotted with squares. The parameters used are $G = 2$, $\gamma = 10$, and $\langle n(0) \rangle = 2$, which implies $R(0) = 0.8$ using Eq. (28).

The master equation can likewise be solved using the initial conditions of Eq. (12) that now read $P_n^1 = -P_n^2 = \delta_{n,2}$ and $P_n^3 = P_n^4 = 0$. Both the analytical formula of Eq. (29) and the numerical solution of the master equation display an exponential decrease of the number of oscillator quanta.

The master equation also allows us to follow the almost stepwise de-excitation of the oscillator as its energy flows away into the bath. This can be seen in Fig. 3, which displays the probability of the oscillator being in the states with two, one, or no quanta—the only states that are encountered along the decay. Initially the oscillator starts with two quanta, and therefore the probability P_2 of being in the $n = 2$ state is unity while $P_1 = P_0 = 0$. At later times, however, as the oscillator loses energy, P_2 decreases in value and eventually reaches zero. The probability P_1 of being in a state with one quantum of excitation rises initially and reaches a maximum but eventually decays as the last remaining quantum leaks into the bath. Finally, as the oscillator reaches steady state by relaxing to its lowest level, P_0 rises in value and eventually reaches one. Note that around the time $t = 2/G$, all three probabilities P_2, P_1, and P_0 are nonzero; quantum mechanically there is a finite probability for the oscillator be in any of these three states.

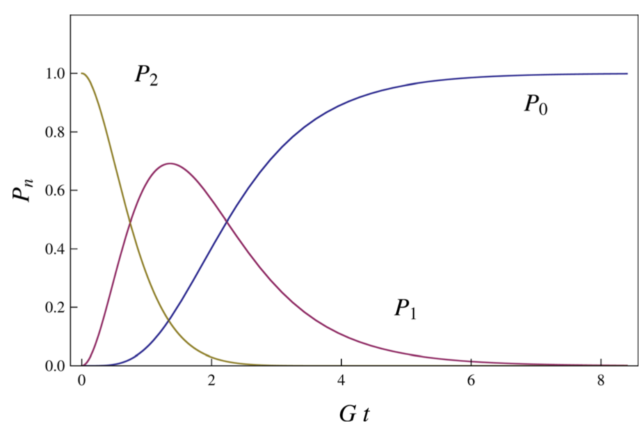

Fig. 3. A graph showing the time evolution of the probability P_n for the oscillator to occupy states $n = 0, 1, 2$ for the situation shown in Fig. 2. The initially occupied $n = 2$ state eventually decays, the $n = 1$ state is transiently populated, and $n = 0$ is ultimately occupied with unit probability.

Equation (30) may be interpreted as the total time required for a quantum of energy to transit from the resonator to the bath. The first term on the RHS corresponds to the time required for the quantum to be transferred from the oscillator to the TLS. The second term denotes the time required for the TLS to spontaneously emit the excitation to the bath. The first term is typically greater than the second as can easily be seen by taking their ratio and using Eq. (10).

How slow or fast is the decay of Eq. (29)? For $\gamma \gg G$, the first term dominates in Eq. (30), giving

$$\Gamma_{\min} \simeq \frac{4G^2}{\gamma}, \tag{31}$$

which can equal zero only in the trivial case of no coupling ($G = 0$). But even if $G \neq 0$, $\Gamma_{\min}$ tends to zero as γ becomes large; however, this limit is not covered by our theory because we have assumed that the coupling to the bath is weak, implying that γ cannot be arbitrarily large [see the discussion above Eq. (10)]. Nevertheless, the fact that Γ decreases with increasing γ does suggest the possibility of long lived quantum states in the presence of large damping, as has been pointed out earlier.[22]

It is straightforward to show that Γ cannot become arbitrarily large. Differentiation of Eq. (30) with respect to γ indicates that the maximum possible value of Γ is given by

$$\Gamma_{\max} = \frac{\gamma}{2} \tag{32}$$

and occurs when $G = \gamma/2$, a relation permitted by Eq. (10).

B. Dry friction: $R(t) \gg 1$

We now consider the regime in which $R(t) \gg 1$. In this case, Eq. (27) implies $\langle \sigma_{11} \rangle_s \simeq 1/2$. This regime corresponds to the situation where the rate at which the TLS absorbs quanta from the oscillator equals the rate at which it is stimulated to emit them, and thus leads to the TLS population dividing itself equally between the ground and excited states. In this case, using $\langle \sigma_{11} \rangle \simeq \langle \sigma_{11} \rangle_s$ in Eq. (20) yields the solution

$$\langle n(t) \rangle = \langle n(0) \rangle - \frac{\gamma t}{2}. \tag{33}$$

The behavior displayed by this simple analytical expression is quite close to that predicted by the full numerical solution of the master equation [Eq. (9)], as can be seen in Fig. 4. This figure shows how the number of quanta in the oscillator decreases with time, starting initially with $\langle n(0) \rangle = 200$ quanta. For this initial excitation of the oscillator, the analytical expression of Eq. (33) is straightforward to plot (we have used $\gamma = 10$ in Fig. 4). The master equation of Eq. (9) has likewise been solved with the initial conditions of Eq. (12) that now read $P_n^1 = -P_n^2 = \delta_{n,200}$ and $P_n^3 = P_n^4 = 0$. Both calculations show a clear linear decrease of the number of oscillator quanta.

It is amusing to note, in the context of the maximum exponential decay rate predicted by Eq. (32), that the maximum linear decay rate predicted by Eq. (33) is also $\gamma/2$. The reader may find it interesting to compare the numerical quantum mechanical energy decay curve of Fig. 4 to the classical solution of the dry friction problem given by Lapidus[9] and to Fig. 5 of Ref. 11.

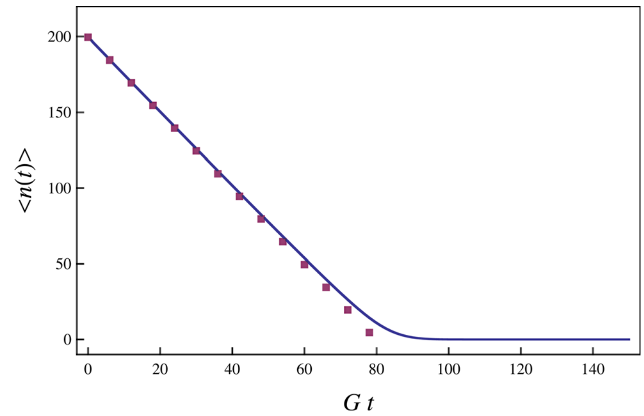

Fig. 4. A graph of oscillator photon number as a function of time in the dry friction regime. The numerical solution of the master equation [Eq. (9)] is shown as a solid curve, while the analytical solution [Eq. (33)] is plotted with squares. The parameters used are $G = 2$, $\gamma = 10$, and $\langle n(0) \rangle = 200$, which implies $R(0) = 8$ using Eq. (28).

C. Crossover regime

In the regime in between fluid and dry friction, we cannot make any approximations to Eq. (27). However, as in the derivation of Eq. (27), we may neglect the slowly varying $d\langle \sigma_{11} \rangle / dt$ term from the LHS of Eq. (20) and use $\langle \sigma_{11} \rangle \simeq \langle \sigma_{11} \rangle_s$ to arrive at the equation

$$\frac{d\langle n \rangle}{dt} = -\frac{\gamma}{2} \left(\frac{\alpha \langle n \rangle}{1 + \alpha \langle n \rangle} \right), \tag{34}$$

where $\alpha = 8(G/\gamma)^2$. An implicit solution to Eq. (34) can be found by inverting the equation to read

$$\frac{dt}{d\langle n \rangle} = -\frac{2}{\gamma} \left(\frac{1 + \alpha \langle n \rangle}{\alpha \langle n \rangle} \right), \tag{35}$$

which is permissible if we avoid situations where the denominator is zero. The solution to Eq. (35), in terms of the initial oscillator photon number $\langle n(0) \rangle$, is given by

$$t = \frac{2}{\gamma} \left[\frac{\gamma^2}{8G^2} \ln \frac{\langle n(0) \rangle}{\langle n(t) \rangle} + (\langle n(0) \rangle - \langle n(t) \rangle) \right]. \tag{36}$$

Equation (36) interpolates smoothly between fluid and dry friction regimes. The first term in the parentheses on the RHS captures the exponential behavior. If the second term is neglected, Eq. (36) can be solved to yield Eq. (29) and the first, dominant contribution to Eq. (30). Similarly, the second term inside the parentheses on the RHS of Eq. (36) describes the linear behavior of Eq. (33). Using Eq. (36), numerical curves can easily be generated and it can also be used to gain some analytical intuition, such as for the time $t_{1/2}$ required for half the quanta to leak away from the oscillator (see Problems 3 and 4 below). Figure 5 demonstrates that Eq. (36) and the numerical solution of Eq. (9) compare quite favorably.

V. SUGGESTED PROBLEMS

In this section, we provide some questions for the reader which are designed to increase familiarity with the physics

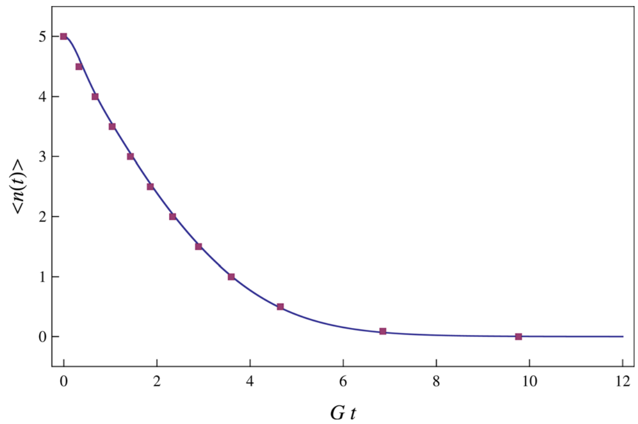

Fig. 5. A graph of oscillator photon number as a function of time in the crossover regime between fluid and dry friction. The numerical solution of the master equation [Eq. (9)] is shown as a solid curve, while the analytical solution [Eq. (36)] is plotted using squares. The parameters used are $G=2$, $\gamma=10$, and $\langle n(0)\rangle=5$, which implies $R(0)=1.26$ using Eq. (28). For short times, the photon decay is linear with time and at later times, the decay becomes exponential.

presented in the sections above and to explore some of its ramifications. The problems only require knowledge of algebra and differential and integral calculus and thus should be amenable to both undergraduate and graduate students.

(1) Derive Eq. (27). Use this equation to make a plot of $\langle\sigma_{11}\rangle_s$ versus R, letting R vary from zero to higher values. Note that $\langle\sigma_{11}\rangle_s$ approaches 1/2 as R takes on large values.

(2) Show that

$$|\langle\sigma_{10}\rangle_s|^2=\frac{1}{2}\left[\frac{R(t)}{1+R(t)^2}\right]^2, \tag{37}$$

which quantifies the coherence internal to the TLS. Notice that the coherence vanishes for very small and very large $R(t)$. Confirm that $|\langle\sigma_{10}\rangle_s|^2$ has a maximum at $R=1$.

(3) Use Eq. (36) to show that $t_{1/2}$, the time required for half the quanta to leak out of the oscillator, is given by

$$t_{1/2}=\frac{1}{\gamma}\left[\left(\frac{\gamma}{2G}\right)^2\ln 2+\langle n(0)\rangle\right]. \tag{38}$$

Generalize this expression to find $t_{1/N}$, where N is an integer. How about $t_{1/e}$?

(4) Plot $1/t_{1/2}$ as a function of $\langle n(0)\rangle$. Discuss how this curve characterizes the loss of quanta from the oscillator for small as well as large $\langle n(0)\rangle$.

VI. DISCUSSION

We have presented a simple system, consisting of a harmonic oscillator and a TLS, which displays fluid and dry friction and may be tuned continuously between the two cases. This is the quantum counterpart to cases of classical friction that are studied in the existing literature. In the classical system, the force of fluid friction increases with the oscillator velocity. In the quantum case, this corresponds to an increased scattering of quanta into the bath, which occurs because the TLS excitation has room to increase from its initially low value. In the classical case, the force of dry friction is independent of the oscillator velocity and remains clamped at a fixed value. In the quantum mechanical case, this clamping is supplied by the saturation of the TLS which acts as a conduit from the oscillator to the reservoir. The analysis presented here should complement other pedagogical studies of the damped quantum mechanical harmonic oscillator.[23,24]

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